MOTION OF A GAS BUBBLE IN A VISCOUS VIBRATING LIQUID

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There are many works ([1-5], for example) containing theoretical results on the motion of a gas bubble in a vibrating liquid. In the present paper we consider the following problem. A closed container is filled with a viscous incompressible liquid in which there is a gas bubble and the container performs a specified periodic translational vibration relative to an inertial rectangular coordinate system, X, Y, Z (the period of the vibrations is T and the container vibrates along the Z axis). The container is deformed in a specified way (compressed and released). The position of the gas bubble relative to the coordinate system X, Y, Z is characterized by the radius vector

$$\mathbf{S} = \frac{1}{Q}\iiint\limits_{\Omega_{XYZ}} \mathbf{R}\,dX\,dY\,dZ,$$

where $\mathbf{R} = (X, Y, Z)$; Ω_{XYZ} is the region occupied by the gas (i.e., the gas bubble), and Q is the volume of the bubble. The flow of the liquid is considered with respect to the coordinate system $X_1 = X - S_X$, $X_2 = Y - S_Y$, $X_3 = Z - S_Z$ (S_X, S_Y, S_Z are the X, Y, Z components of the vector $\mathbf{S}$). The smallest distance from the gas bubble to the walls of the container is large in comparison with the largest dimension of the bubble and hence the walls of the container can be assumed to be infinitely distant from the bubble. The velocity $\mathbf{V}$ of the liquid satisfies the condition

$$\mathbf{V} \sim \tilde{U}\mathbf{k} - d\mathbf{S}/dt\ ; \qquad X_1^2 + X_2^2 + X_3^2 \to \infty,$$

where t is the time, $\tilde{U} = \mathrm{Real}\sum\limits_{m=1}^{\infty} U_m e^{2m\pi i t/T}$ (U_m are constants), $\mathbf{k} = (0, 0, 1)$. The pressure P of the liquid then satisfies the condition

$$P \sim -\rho\,(d\tilde{U}/dt)\,X_3 + \tilde{P}\ ; \qquad X_1^2 + X_2^2 + X_3^2 \to \infty,$$

where ρ is the density of the liquid and $\tilde{P}$ is a function of t. The dependence of $\tilde{P}$ on t is determined by the manner in which the container is deformed. We assume that

$$\tilde{P} = P_0 + \mathrm{Real}\sum_{m=1}^{\infty} P_m e^{2m\pi i t/T}$$

(P_0, P_m are constants). The flow of the liquid is steady-state (i.e., it does not depend on the initial conditions). In the absence of vibrations and deformation of the container [$U_m = P_m = 0$ (m = 1, 2, ...)] the gas bubble will be a sphere $\sqrt{X_1^2 + X_2^2 + X_3^2} \leqslant A_0$, $\mathbf{V} = 0$, $P = P_0$. The pressure P_g and the volume of the gas are connected by the adiabatic equation

$$P_g Q^\gamma = P_{g0} Q_0^\gamma,$$

where γ is the adiabatic index, $P_{g0} = P_0 + 2\sigma/A_0$ (σ is the surface tension), $Q_0 = (4\pi/3)\,A_0^3$. It is required to find the motion of the gas bubble with respect to the coordinate system X, Y, Z, i.e., to find $\mathbf{S}$ as a function of time.

1. Let $\tau = t/T$; $x_1 = X_1/A_0$; $x_2 = X_2/A_0$; $x_3 = X_3/A_0$; $\mathbf{r} = (x_1, x_2, x_3)$; $r = |\mathbf{r}|$; Γ is the surface bounding the region $\Omega_{x_1x_2x_3}$ occupied by the gas (the free boundary of the region occupied by the liquid); H is the mean curvature of Γ; $\eta = A_0H - 1$; $\mathbf{n}$ is the outward unit normal to Γ; Ξ is the velocity of Γ in the direction $\mathbf{n}$; $\xi = T\Xi/A_0$; $\mathbf{v} = (v_i) = T\mathbf{V}/A_0$; $p = T^2(P - P_0)/(\rho A_0^2)$; $\mathbf{w} = (1/A_0)d\mathbf{S}/d\tau$; ν is the kinematic viscosity of the liquid; $\mathrm{Re} = A_0^2/(\nu T)$

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is the Reynolds number, $\mathbf{P}$ is the stress tensor in the liquid; $\mathbf{I} = (I_{ij})$ is the unit tensor; $\mathbf{p} = (p_{ij}) = T^2 (\mathbf{P} + P_0\mathbf{I}) / (\rho A_0{}^2)$ $[p_{iv} = -pI_{ij} + (1/\mathrm{Re})(\partial v_i/\partial x_j + \partial v_j/\partial x_i)]$; $\hat{P}$ is the largest value of $|\widetilde{P} - P_0|$; $\widetilde{p} = (\widetilde{P} - P_0)/\hat{P} = \mathrm{Real} \sum_{m=1}^{\infty} p_m e^{2m\pi i\tau}$; $\hat{U}$ is the largest value of $|\widetilde{U}|$; $\widetilde{u} = \widetilde{U}/\hat{U} = \mathrm{Real} \sum_{m=1}^{\infty} u_m e^{2m\pi i\tau}$; $\varepsilon = \hat{U}T/A_0$; $\varkappa = \hat{P}T^2/(\rho A_0{}^2)$; $\lambda = \sigma T^2/(\rho A_0{}^3)$; $\mu = P_{g0}T^2/(\rho A_0{}^2)$; $p_g = T^2(P_g - P_{g0})/(\rho A_0{}^2) = \mu(Q_0{}^\gamma/Q^\gamma - 1)$.

The equation of the surface Γ, the Navier–Stokes equations, the equation of continuity, and the conditions which must be satisfied on Γ and in the limit $r \to \infty$ can be written as

$$\chi = 0$$
$$(\chi < 0 \text{ inside } \Omega_{x_1x_2x_3},\ \chi > 0 \text{ outside } \Omega_{x_1x_2x_3}); \tag{1.1}$$

$$\frac{\partial \mathbf{v}}{\partial \tau} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p - \frac{1}{\mathrm{Re}}\Delta\mathbf{v} + \frac{d\mathbf{w}}{d\tau} = 0; \tag{1.2}$$

$$\nabla\cdot\mathbf{v} = 0; \tag{1.3}$$

$$\mathbf{n}\cdot\mathbf{v} - \xi = 0,\quad \mathbf{n}\cdot\mathbf{p} + (p_\mathrm{r} - 2\lambda\eta)\mathbf{n} = 0 \text{ on } \Gamma; \tag{1.4}$$

$$\mathbf{v} \sim \varepsilon\widetilde{u}\mathbf{k} - \mathbf{w},\quad p \sim -\varepsilon\frac{d\widetilde{u}}{d\tau}x_3 + \varkappa\widetilde{p} \quad \text{for} \quad r \to \infty. \tag{1.5}$$

The following equation must also be satisfied

$$\iiint_{\Omega_{x_1x_2x_3}} \mathbf{r}\,dx_1dx_2dx_3 = 0. \tag{1.6}$$

It is necessary to find the solution of (1.1)-(1.6) for χ, $\mathbf{v}$, p, $\mathbf{w}$ in order to determine the dependence of S on t.

2. We will consider the problem (1.1)-(1.6) when ε is small in comparison with unity. We assume that in the limit $\varepsilon \to 0$

$$\begin{gathered}\chi \sim \chi^{(0)} + \varepsilon\chi^{(1)},\quad \mathbf{v} \sim \mathbf{v}^{(0)} + \varepsilon\mathbf{v}^{(1)},\\ p \sim p^{(0)} + \varepsilon p^{(1)},\quad \mathbf{w} \sim \mathbf{w}^{(0)} + \varepsilon\mathbf{w}^{(1)}.\end{gathered} \tag{2.1}$$

From (1.1)-(1.6) and (2.1), we have in the M-th approximation (M = 0, 1)

$$\chi^{(0)} + M\varepsilon\chi^{(1)} = 0 \tag{2.2}$$

for the equation of the surface $\Gamma^{(M)}$ bounding the region $\Omega^{(M)}$ occupied by the gas;

$$\frac{\partial \mathbf{v}^{(M)}}{\partial \tau} + (\mathbf{v}^{(0)}\cdot\nabla)\mathbf{v}^{(M)} + M(\mathbf{v}^{(1)}\cdot\nabla)\mathbf{v}^{(0)} + \nabla p^{(M)} - \frac{1}{\mathrm{Re}}\Delta\mathbf{v}^{(M)} + \frac{d\mathbf{w}^{(M)}}{d\tau} = 0; \tag{2.3}$$
$$\nabla\cdot\mathbf{v}^{(M)} = 0; \tag{2.4}$$
$$\lim_{\varepsilon\to 0}\left[\varepsilon^{-M}\left(\mathbf{n}^{(M)}\cdot\mathbf{v} - \xi^{(M)}\right)\big|_{\Gamma^{(M)}}\right] = 0, \tag{2.5}$$
$$\lim_{\varepsilon\to 0}\left\{\varepsilon^{-M}\left[\mathbf{n}^{(M)}\cdot\mathbf{p} + \left(p_\mathrm{r}^{(M)} - 2\lambda\eta^{(M)}\right)\mathbf{n}^{(M)}\right]\big|_{\Gamma^{(M)}}\right\} = 0;$$

$$\mathbf{v}^{(M)} \sim M\widetilde{u}\mathbf{k} - \mathbf{w}^{(M)},\quad p^{(M)} \sim (1-M)\varkappa\widetilde{p} - M\frac{d\widetilde{u}}{d\tau}x_3 \quad \text{for} \quad r \to \infty; \tag{2.6}$$

$$\iiint_{\Omega^{(M)}} \mathbf{r}\,dx_1dx_2dx_3 = 0, \tag{2.7}$$

where $\mathbf{n}^{(M)}$, $\eta^{(M)}$, $\xi^{(M)}$, $p_g{}^{(M)}$ are the quantities for $\mathbf{n}$, η, ξ, p_g for $\Gamma = \Gamma^{(M)}$.

Let M = 0. In the zeroth approximation the gas bubble is a sphere $r \leq 1 + a$, whose center is at rest with respect to the coordinate system X, Y, Z. The flow of the liquid is symmetric to the origin of the coordinate system x_1, x_2, x_3. Hence

$$\chi^{(0)} = r - 1 - a; \tag{2.8}$$

$$\mathbf{w}^{(0)} = 0; \tag{2.9}$$

$$\partial v_r^{(0)}/\partial\theta = 0, \quad \partial v_r^{(0)}/\partial\varphi = 0; \tag{2.10}$$

$$v_\theta^{(0)} = 0, \quad v_\varphi^{(0)} = 0, \tag{2.11}$$

where r, θ, φ are spherical coordinates [θ is the angle between the vectors (0, 0, 1) and (x_1, x_2, x_3) ($0 \leq \theta \leq \pi$); φ is the angle between the vectors (1, 0, 0) and (x_1, x_2, 0) ($0 \leq \varphi < 2\pi$)]; $v_r^{(0)}$, $v_\theta^{(0)}$, $v_\varphi^{(0)}$ are the r, θ, φ components of the vector $\mathbf{v}^{(0)}$.

The relation (2.7) will be satisfied for any positive value of $1 + a$. It follows from (2.2)-(2.6) and (2.8)-(2.11) that

$$v_r^{(0)} = (1+a)^2 (da/d\tau)/r^2; \tag{2.12}$$

$$p^{(0)} = \varkappa\tilde{p} + \frac{(1+a)^2}{r}\left\{\frac{d^2a}{d\tau^2} + \frac{1}{1+a}\left(\frac{da}{d\tau}\right)^2\left[2 - \frac{(1+a)^3}{2r^3}\right]\right\}; \tag{2.13}$$

$$\frac{d^2a}{d\tau^2} + \frac{1}{1+a}\left\{\frac{3}{2}\left(\frac{da}{d\tau}\right)^2 + \frac{4}{\mathrm{Re}\,(1+a)}\frac{da}{d\tau} - 2\lambda\frac{a}{1+a} + \mu\left[1-(1+a)^{-3\gamma}\right] + \varkappa\tilde{p}\right\} = 0. \tag{2.14}$$

3. Let M = 1. We will consider the problem (2.2)-(2.7) when ϰ is small in comparison with unity. It follows from (2.8) and (2.11)-(2.14) that in the limit $\varkappa \to 0$

$$\chi^{(0)} \sim \chi_{(0)}^{(0)} + \varkappa\chi_{(1)}^{(0)}, \quad \mathbf{v}^{(0)} \sim \varkappa\mathbf{v}_{(1)}^{(0)}, \quad p^{(0)} \sim \varkappa p_{(1)}^{(0)}, \tag{3.1}$$

where $\chi_{(0)}^{(0)} = r - 1$; $\chi_{(1)}^{(0)} = \mathrm{Re\,Real}\sum_{m=1}^{\infty}\frac{p_m}{(3\gamma\mu - 2\lambda - 4m^2\pi^2)\,\mathrm{Re} + 8m\pi i}e^{2m\pi i\tau}$; $\mathbf{v}_{(1)}^{(0)} = -\left(d\chi_{(1)}^{(0)}/d\tau\right)\mathbf{r}/r^3$; $p_{(1)}^{(0)} = \tilde{p} - (d^2\chi_{(1)}^{(0)}/d\tau^2)/r$. The equations $\chi_{(0)}^{(0)} = 0$, $\chi_{(0)}^{(0)} + \varkappa\chi_{(1)}^{(0)} = 0$ determine the surfaces $\Gamma_{(0)}^{(0)}$ and $\Gamma_{(1)}^{(0)}$, respectively. We assume that when $\varkappa \to 0$

$$\chi^{(1)} \sim \chi_{(0)}^{(1)} + \varkappa\chi_{(1)}^{(1)}, \quad \mathbf{v}^{(1)} \sim \mathbf{v}_{(0)}^{(1)} + \varkappa\mathbf{v}_{(1)}^{(1)}, \quad p^{(1)} \sim p_{(0)}^{(1)} + \varkappa p_{(1)}^{(1)}, \quad \mathbf{w}^{(1)} \sim \mathbf{w}_{(0)}^{(1)} + \varkappa\mathbf{w}_{(1)}^{(1)}. \tag{3.2}$$

According to (2.2)-(2.7), (3.1), and (3.2), in the N-th approximation (N = 0, 1), we have

$$\chi_{(0)}^{(0)} + \varepsilon\chi_{(0)}^{(1)} + N\varkappa\left(\chi_{(1)}^{(0)} + \varepsilon\chi_{(1)}^{(1)}\right) = 0 \tag{3.3}$$

for the equation of the surface $\Gamma_{(N)}^{(1)}$ bounding the region $\Omega_{(N)}^{(1)}$ occupied by the gas;

$$\frac{\partial\mathbf{v}_{(N)}^{(1)}}{\partial\tau} + \nabla p_{(N)}^{(1)} - \frac{1}{\mathrm{Re}}\Delta\mathbf{v}_{(N)}^{(1)} + \frac{d\mathbf{w}_{(N)}^{(1)}}{d\tau} = -N\left[\left(\mathbf{v}_{(1)}^{(0)}\cdot\nabla\right)\mathbf{v}_{(0)}^{(1)} + \left(\mathbf{v}_{(0)}^{(1)}\cdot\nabla\right)\mathbf{v}_{(1)}^{(0)}\right]; \tag{3.4}$$

$$\nabla\cdot\mathbf{v}_{(N)}^{(1)} = 0; \tag{3.5}$$

$$\lim_{\varkappa\to 0}\lim_{\varepsilon\to 0}\left[\varkappa^{-N}\varepsilon^{-1}\left(\mathbf{n}_{(N)}^{(1)}\cdot\mathbf{v} - \xi_{(N)}^{(1)}\right)\Big|_{\Gamma_{(N)}^{(1)}}\right] = 0, \tag{3.6}$$

$$\lim_{\varkappa\to 0}\lim_{\varepsilon\to 0}\left\{\varkappa^{-N}\varepsilon^{-1}\left[\mathbf{n}_{(N)}^{(1)}\cdot\mathbf{p} + \left(p_{\Gamma(N)}^{(1)} - 2\lambda\eta_{(N)}^{(1)}\right)\mathbf{n}_{(N)}^{(1)}\right]\Big|_{\Gamma_{(N)}^{(1)}}\right\} = 0;$$

$$\mathbf{v}_{(N)}^{(1)} \sim (1-N)\left(\tilde{u}\mathbf{k} - \mathbf{w}_{(0)}^{(1)}\right) - N\mathbf{w}_{(1)}^{(1)}, \quad p_{(N)}^{(1)} \sim (N-1)\frac{d\tilde{u}}{d\tau}x_3 \quad \text{as} \quad r \to \infty; \tag{3.7}$$

$$\iiint\limits_{\Omega_{(N)}^{(1)}} \mathbf{r}\, dx_1 dx_2 dx_3 = 0, \tag{3.8}$$

where $\mathbf{n}_{(N)}{}^{(1)}$, $\eta_{(N)}{}^{(1)}$, $\xi_{(N)}{}^{(1)}$, $p_{g(N)}{}^{(1)}$ are the quantities for $\mathbf{n}$, η, ξ, p_g for $\Gamma = \Gamma_{(N)}{}^{(1)}$.

Let N = 0. We assume that $\chi_{(0)}{}^{(1)} \equiv 0$. Then $\Gamma_{(0)}{}^{(1)} = \Gamma_{(0)}{}^{(0)}$, (3.8) is satisfied, and (3.6) reduces to the conditions

$$v_{(0)r}^{(1)} = 0, \quad -p_{(0)}^{(1)} + \frac{2}{\mathrm{Re}} \frac{\partial v_{(0)r}^{(1)}}{\partial r} = 0,$$
$$\partial v_{(0)\theta}^{(1)}/\partial r - v_{(0)\theta}^{(1)} = 0, \quad \partial v_{(0)\varphi}^{(1)}/\partial r - v_{(0)\varphi}^{(1)} = 0 \quad \text{on } \Gamma_{(0)}^{(0)}, \tag{3.9}$$

where $v_{(0)r}{}^{(1)}$, $v_{(0)\theta}{}^{(1)}$, $v_{(0)\varphi}{}^{(1)}$ are the r, θ, φ components of the vector $\mathbf{v}_{(0)}{}^{(1)}$. The The problem (3.4), (3.5), (3.7), (3.9) has the solution

$$v_{(0)r}^{(1)} = \frac{1}{r^2 \sin\theta} \frac{\partial\psi}{\partial\theta}, \quad v_{(0)\theta}^{(1)} = -\frac{1}{r \sin\theta} \frac{\partial\psi}{\partial r}, \tag{3.10}$$

$$v_{(0)\varphi}^{(1)} = 0, \quad p_{(0)}^{(1)} = \left\{\left[-\frac{\partial^2}{\partial\tau\partial r} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^3}{\partial r^3} - \frac{2}{r^2}\frac{\partial}{\partial r} + \frac{4}{r^3}\right)\right]\psi - \frac{dw}{d\tau} r \sin^2\theta\right\} \frac{\cos\theta}{\sin^2\theta};$$
$$\mathbf{w}_{(0)}^{(1)} = w\mathbf{k}, \tag{3.11}$$

where

$$\psi = \left[\frac{1}{2}(\tilde{u} - w) r^2 + \frac{1}{3}\frac{w}{r} - 6\,\mathrm{Real} \sum_{m=1}^{\infty} \frac{u_m}{q_m^3 + 3q_m^2 + 18q_m + 18}\left(q_m + \frac{1}{r}\right) e^{q_m(1-r)} \times \right.$$
$$\left. \times\, e^{2m\pi i\tau}\right] \sin^2\theta; \quad w = \mathrm{Real} \sum_{m=1}^{\infty} w_m e^{2m\pi i\tau} \left(w_m = 3u_m (q_m^3 + 3q_m^2 + \right.$$
$$\left. + 6q_m + 6)/(q_m^3 + 3q_m^2 + 18q_m + 18); \quad q_m = (1 + i)\sqrt{m\pi\,\mathrm{Re}}\right).$$

Hence, $\chi_{(0)}{}^{(1)} \equiv 0$ and the solution (3.10), (3.11) of the problem (3.4), (3.5), (3.7), (3.9) is also the solution of the problem (3.3)-(3.8).

Let N = 1. We assume that $\chi_{(1)}{}^{(1)} \equiv 0$. Then $\Gamma_{(1)}{}^{(1)} = \Gamma_{(1)}{}^{(0)}$, (3.8) is satisfied, and (3.6) reduces to the conditions

$$v_{(1)r}^{(1)} = \chi_{(1)}^{(0)} \frac{\partial v_{(0)r}^{(1)}}{\partial r},$$
$$-p_{(1)}^{(1)} + \frac{2}{\mathrm{Re}}\frac{\partial v_{(1)r}^{(1)}}{\partial r} = \chi_{(1)}^{(0)} \frac{\partial}{\partial r}\left(-p_{(0)}^{(1)} + \frac{2}{\mathrm{Re}}\frac{\partial v_{(0)r}^{(1)}}{\partial r}\right),$$
$$\frac{\partial v_{(1)r}^{(1)}}{\partial\theta} + \frac{\partial v_{(1)\theta}^{(1)}}{\partial r} - v_{(1)\theta}^{(1)} = \chi_{(1)}^{(0)} \frac{\partial}{\partial r}\left(\frac{\partial v_{(0)r}^{(1)}}{\partial\theta} + \frac{\partial v_{(0)\theta}^{(1)}}{\partial r} - \frac{v_{(0)\theta}^{(1)}}{r}\right), \quad \frac{1}{\sin\theta}\frac{\partial v_{(1)r}^{(1)}}{\partial\varphi} + \tag{3.12}$$
$$+ \frac{\partial v_{(1)\varphi}^{(1)}}{\partial r} - v_{(1)\varphi}^{(1)} = 0 \quad \text{on } \Gamma_{(0)}^{(0)},$$

where $v_{(1)r}{}^{(1)}$, $v_{(1)\theta}{}^{(1)}$, $v_{(1)\varphi}{}^{(1)}$ are the r, θ, φ components of the vector $\mathbf{v}_{(1)}{}^{(1)}$. The problem (3.4), (3.5), (3.7), (3.12) has the solution

$$v_{(1)r}^{(1)} = \frac{1}{r^2 \sin\theta}\frac{\partial\psi'}{\partial\theta}, \quad v_{(1)\theta}^{(1)} = -\frac{1}{r\sin\theta}\frac{\partial\psi'}{\partial r},$$
$$v_{(1)\varphi}^{(1)} = 0, \quad p_{(1)}^{(1)} = \left\{\left[-\frac{\partial^2}{\partial\tau\partial r} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^3}{\partial r^3} - \frac{2}{r^2}\frac{\partial}{\partial r} + \frac{4}{r^3}\right)\right]\psi' - \frac{dw'}{d\tau} r\sin^2\theta - v_{(1)r}^{(0)}\frac{\partial^2\psi}{\partial r^2}\right\}\frac{\cos\theta}{\sin^2\theta}; \tag{3.13}$$

$$\mathbf{w}_{(1)}^{(1)} = (\overline{w} + w')\,\mathbf{k}, \tag{3.14}$$

where

$$\psi' = \left\{-\frac{1}{2}(\overline{w} + w') r^2 + \frac{\alpha_0}{r} + \beta_0 r + \Phi_0 + \mathrm{Real}\sum_{m=1}^{\infty}\left[\frac{\alpha_m}{r} + \right.\right.$$

$$+\beta_m\left(q_m+\frac{1}{r}\right)e^{-q_m r}+\Phi_m\Big]e^{2m\pi i\tau}\Big\}\sin^2\theta;\quad \overline{w}=\frac{1}{2}\operatorname{Re}\operatorname{Real}\sum_{m=1}^{\infty}p_m^* u_m\times$$

$$\times\frac{q_m^2\left[4\left(q_m^2+3q_m+3\right)+q_m^4 e^{q_m}\int\limits_1^{\infty}\frac{1}{r}e^{-q_m r}dr\right]}{\left[(4m^2\pi^2-3\gamma\mu+2\lambda)\operatorname{Re}+8m\pi i\right]\left(q_m^3+3q_m^2+18q_m+18\right)};$$

$$w'=\operatorname{Real}\sum_{m=1}^{\infty}w_m' e^{2m\pi i\tau}$$

(p_m^* are the complex conjugates of the constants p_m; w_m', α_0, α_m, β_0, β_m and Φ_0, Φ_m are constants and functions of r, respectively, and are given by the relations

$$\alpha_0=\frac{1}{6}\left(\int\limits_0^1 G\,d\tau-\varphi_0|_{r=1}\right),\ \beta_0=\frac{1}{6}\int\limits_0^1(F-2G)\,d\tau,\ w_m'-2\alpha_m-2(q_m+1)e^{-q_m}\beta_m=$$

$$=2\left(\int\limits_0^1 E e^{-2m\pi i\tau}d\tau+\Phi_m|_{r=1}\right),\ 4w_m'+\left(q_m^2+4\right)\alpha_m+4\left(q_m^2+q_m+1\right)e^{-q_m}\beta_m=$$

$$=2\int\limits_0^1(4E+F)e^{-2m\pi i\tau}d\tau+\frac{\left(q_m^2+1\right)\operatorname{th}q_m-q_m}{\operatorname{th}q_m-q_m}\varphi_m|_{r=1}+2\left(q_m^2+4\right)\Phi_m|_{r=1},$$

$$2w_m'+2\alpha_m+\left(q_m^3+3q_m^2+2q_m+2\right)e^{-q_m}\beta_m=2\int\limits_0^1(2E+G)e^{-2m\pi i\tau}d\tau-\varphi_m|_{r=1}+4\Phi_m|_{r=1},$$

$$E=-\frac{\chi_{(1)}^{(0)}}{\cos\theta}\left(\frac{\partial}{\partial r}v_{(0)r}^{(1)}\right)\Big|_{r=1},$$

$$F=\operatorname{Re}\left\{\frac{\partial}{\partial r}\left[-\frac{\chi_{(1)}^{(0)}}{\cos\theta}\left(-p_{(0)}^{(1)}+\frac{2}{\operatorname{Re}}\frac{\partial v_{(0)r}^{(1)}}{\partial r}\right)+\frac{d\chi_{(1)}^{(0)}/d\tau}{\sin\theta}rv_{(0)\theta}^{(1)}\right]\right\}\Big|_{r=1},$$

$$G=-\frac{\chi_{(1)}^{(0)}}{\sin\theta}\left[\frac{\partial}{\partial r}\left(\frac{\partial v_{(0)r}^{(1)}}{\partial\theta}+\frac{\partial v_{(0)\theta}^{(1)}}{\partial r}-\frac{v_{(0)\theta}^{(1)}}{r}\right)\right]\Big|_{r=1},$$

$$\Phi_0=-\frac{1}{3}\left(\frac{1}{r}\int\limits_1^r\varphi_0 r^2 dr+r^2\int\limits_r^{\infty}\frac{\varphi_0}{r}\,dr\right),$$

$$\Phi_m=-\frac{1}{3}\left(\frac{1}{r}\int\limits_1^r\varphi_m r^2 dr+r^2\int\limits_r^{\infty}\frac{\varphi_m}{r}dr\right),$$

$$\varphi_0=-\frac{1}{3}\operatorname{Re}\left(\frac{1}{r}\int\limits_1^r H_0 r^3 dr+r^2\int\limits_r^{\infty}H_0 dr\right),$$

$$\varphi_m=\frac{\operatorname{Re}}{q_m^3 r}\left[(q_m r+1)e^{-q_m r}\int\limits_1^r H_m(\operatorname{sh}q_m r-q_m r\operatorname{ch}q_m r)\,dr+(\operatorname{sh}q_m r-q_m r\operatorname{ch}q_m r)\int\limits_r^{\infty}H_m(q_m r+1)e^{-q_m r}dr\right],$$

$$H_0=\int\limits_0^1 H\,d\tau,\quad H_m=2\int\limits_0^1 He^{-2m\pi i\tau}d\tau,$$

$$H=-\frac{d\chi_{(1)}^{(0)}/d\tau}{r^3\sin^2\theta}\left(\frac{\partial^3}{\partial r^3}-\frac{2}{r}\frac{\partial^2}{\partial r^2}-\frac{2}{r^2}\frac{\partial}{\partial r}+\frac{8}{r^3}\right)\psi).$$

Hence, $\chi_{(1)}^{(1)}\equiv 0$ and the solution (3.13), (3.14) of the problem (3.4), (3.5), (3.7), (3.12) is also the solution of the problem (3.3)-(3.8).

4. Let ε and ϰ be small in comparison with unity and suppose that ε is small in comparison with ϰ. Then the relations

$$\chi=\chi^{(0)};$$
$$\mathbf{v}=\mathbf{v}^{(0)}+\varepsilon\mathbf{v}_{(0)}^{(1)}+\varepsilon\varkappa\mathbf{v}_{(1)}^{(1)};\quad p=p^{(0)}+\varepsilon p_{(0)}^{(1)}+\varepsilon\varkappa p_{(1)}^{(1)};\tag{4.1}$$

$$w = \varepsilon w_{(0)}^{(1)} + \varepsilon \varkappa w_{(1)}^{(1)} \quad (4.2)$$

along with (2.8), (2.11)-(2.14), (3.10), (3.11), (3.13), (3.14) determine the approximate solution of the problem (1.1)-(1.6). The solution [with use of (1.1)] satisfies (1.3), (1.5), and (1.6) exactly and (1.2) and (1.4) approximately, to within terms small in comparison with $\varepsilon\varkappa$.

Note that according to (1.1), (2.8), and (4.1), the gas bubble is a sphere and S is the position vector to the center of the bubble. Using (3.11), (3.14), and (4.2), we find

$$\mathbf{S} = \left(\mathrm{Real} \sum_{m=1}^{\infty} S_m e^{2m\pi it/T} + \overline{W}t\right)\mathbf{k} + \mathbf{S}_0, \quad (4.3)$$

where $S_m = A_0\varepsilon(w_m + \varkappa w_m')/(2m\pi i)$; $\overline{W} = (A_0/T)\varepsilon\varkappa\overline{w}$; S_0 is constant. The dependence of S on t is determined approximately by (4.3). In particular, it follows from (4.3) that the gas bubble moves along a straight line parallel to the Z axis and its motion is composed of a vibration and a displacement in the direction **k** (for $\overline{W} > 0$) or $-\mathbf{k}$ (for $\overline{W} < 0$). Hence, vibrations of the liquid (time variation of the velocity and pressure of the liquid) can induce a nonzero average displacement of the bubble. The cause of this displacement is the fact that the conditions for motion of the bubble up and down the axis of vibration of the container are not identical.

LITERATURE CITED

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FLOW STRUCTURE OF A ROTATING LIQUID AFTER MOTION OF A BODY IN IT

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We report the results of an experimental study of the flow structure of a column of liquid which is initially rotating rigidly, after a body is pulled through it in a direction parallel to the axis of rotation. It is shown that the general qualitative result of the motion of the body through the rotating liquid is the formation of a system of cyclone and anticyclone vortices with oscillatory motion of the liquid in them. The properties of these vortices match those reported in [1].

The experimental apparatus is shown schematically in Fig. 1. A transparent vertical cylindrical container 5, in which a liquid rotates with a constant angular velocity ω. The motion of the initially rigidly rotating liquid is perturbed by one or several bodies 6 which rotate with the container and complete one pass through the liquid from the bottom of the container to the free surface of the liquid 3 in a direction parallel to the axis of rotation. Thin (0.5 mm) plates in the shape of a circle or a section of a circle were used as the bodies. The plates were mounted parallel to the bottom on thin rods 4 of identical length, which were attached to the disk 2 above the free surface of the liquid. The disk was rotated together with the container and was displaced upward with the help of the rod 1. Until the

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